Proof Without Words: Ptolemy’s Inequality

Claudi Alsina
Universitat Politècnica de Catalunya

Roger B. Nelsen
Lewis & Clark College

Ptolemy’s Inequality. In a convex quadrilateral with sides of length \(a, b, c, d\) (in that order) and diagonals of length \(p\) and \(q\), we have \(pq \leq ac + bd\).

Proof.

Note. The angle at the top of the figure, \(\delta_2 + \beta_1 + \beta_2 + \delta_1\), is drawn as being smaller than \(\pi\), but the broken line representing \(ac + bd\) is at least as long as the base of the parallelogram in any case. In a cyclic quadrilateral, pairs of opposite sides have sum \(\pi\) so that \(\delta_2 + \beta_1 + \beta_2 + \delta_1 = \pi\), leading to equality:

Ptolemy’s Theorem. In a cyclic quadrilateral with sides of length \(a, b, c, d\) (in that order) and diagonals of length \(p\) and \(q\), we have \(pq = ac + bd\).

Summary. Ptolemy: In a convex quadrilateral with sides of length \(a, b, c, d\) (in that order) and diagonals of length \(p\) and \(q\), we have \(pq \leq ac + bd\).