How close are pairwise and mutual independence?

Roger B. Nelsen, Manuel Úbeda-Flores

Article Info

Article history:
Received 31 March 2012
Received in revised form 5 June 2012
Accepted 6 June 2012
Available online 13 June 2012

MSC:
primary 62H05
secondary 62E10

Keywords:
Copula
Exchangeability
Mutual independence
Pairwise independence
Quasi-copula

1. Introduction

The important distinction between pairwise independence and mutual independence of a set of $n \geq 3$ events or random variables is one that is sometimes difficult to grasp. Indeed, when speaking of a set of three events that are pairwise but not mutually independent, Feller (1968) observes that “such occurrences are so rare that their possibility passed unnoticed until S. Bernstein constructed an artificial example”. For a thorough treatment of Bernstein’s example, see Derriennic and Klopotowski (2000). Many textbooks, when discussing the difference between pairwise and mutual independence, present only an example of a set of events or random variables which are pairwise but not mutually independent. Intuitively, it seems reasonable that the joint distribution of $n$ random variables which are pairwise independent is in some sense “close” to the joint distribution of $n$ mutually independent random variables with the same marginal distributions, and that the joint distributions should become “closer” as $n$ increases. After presenting some background material about copulas in the next section, we examine in detail the trivariate case in Section 3. Using some known results on bounds on sets of copulas with certain properties, we examine the joint distribution of three random variables when one pair, two pairs, or all three pairs are independent. In the following section, we present some asymptotic results in the general case. We conclude with some comments about the weaker notions of pairwise and mutual exchangeability.

2. Preliminaries

Let $X = (X_1, X_2, \ldots, X_n)$ be a vector of real-valued continuous random variables. Since the notions of pairwise and mutual independence depend on relationships among the distribution functions of $X_1, X_2, \ldots, X_n$ and not on the distribution

* Corresponding author.
E-mail addresses: nelsen@lclark.edu (R.B. Nelsen), mubeda@ual.es (M. Úbeda-Flores).
functions themselves, the use of copulas will simplify matters. For any natural number \( n \geq 2 \), an \( n \)-dimensional copula (for short, an \( n \)-copula) is a function \( C : [0, 1]^n \rightarrow [0, 1] \) that satisfies:

1. **Boundary conditions:** For every \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) in \([0, 1]^n\), \( C(\mathbf{u}) = 0 \) if at least one coordinate of \( \mathbf{u} \) is equal to 0; and \( C(\mathbf{u}) = 1 \) if all the coordinates of \( \mathbf{u} \) are 1 except perhaps \( u_i \); and

2. **The \( n \)-increasing property:** \( \forall i \neq j \) for every \( n \)-box \( B = \prod_{i=1}^n [a_i, b_i] \) in \([0, 1]^n\), where the sum is taken over all the vertices \( v = (v_1, v_2, \ldots, v_n) \) of \( B \) (i.e., each \( v_i \) is equal to either \( a_k \) or \( b_k \) and \( \lambda(\mathbf{v}) \) is the number of indices \( k \) such that \( v_k = a_k \); note that \( C(\mathbf{u}) = V_C(\prod_{i=1}^n [0, u_i]) \).

We will refer to \( V_C(B) \) as the \( C \)-volume of \( B \); and in the sequel we will consider the \( C \)-volume of an \( n \)-box for real-valued functions on \([0, 1]^n\) which may not be copulas.

The importance of copulas in probability and statistics is partially explained by Sklar’s theorem (Sklar, 1959): Let \( \mathbf{X} = (X_1, X_2, \ldots, X_n) \) be a random vector with joint distribution function \( H \) and one-dimensional marginal distributions \( \mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n \) respectively. Then there exists an \( n \)-copula \( C \) such that

\[
H(x) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) \quad \forall \mathbf{x} = (x_1, x_2, \ldots, x_n) \in [-\infty, \infty]^n.
\]

Thus copulas link joint distribution functions to their one-dimensional marginals. When the components of \( \mathbf{X} \) are continuous, the copula is unique and is called the copula of \( \mathbf{X} \). As a consequence of Sklar’s theorem, to measure the “distance” between two joint distribution functions with the same marginal distributions, we need only define a distance between the corresponding copulas.

Some results about copulas important for our study are the following:

1. An \( n \)-copula \( C \) is the restriction to \([0, 1]^n\) of an \( n \)-dimensional distribution function whose one-dimensional marginals are uniform on \([0, 1] \).

2. As a consequence of the \( n \)-increasing property, every \( n \)-copula \( C \) is nondecreasing and Lipschitz in each argument; that is, for each \( k \in \{1, 2, \ldots, n\} \), \( u_k \leq u_k' \) implies \( C(u_1, \ldots, u_k, \ldots, u_n) \leq C(u_1, \ldots, u_k', \ldots, u_n) \) and \( C(u_1, \ldots, u_k', \ldots, u_n) - C(u_1, \ldots, u_k, \ldots, u_n) \leq \lambda_k - \lambda_k' \).

3. The \( k \)-dimensional marginals (or \( k \)-margins) of an \( n \)-copula \( \mathbf{C} \), \( 2 \leq k < n \), are the \( k \)-variate functions on \([0, 1]^k \) obtained by setting \( n - k \) of the arguments of \( \mathbf{C} \) equal to 1. Every \( k \)-margin of an \( n \)-copula is a \( k \)-copula.

4. For any \( n \)-copula \( \mathbf{C} \) and all \( \mathbf{u} \) in \([0, 1]^n \), we have

\[
W^n(\mathbf{u}) = \max(u_1 + u_2 + \cdots + u_n - n + 1, 0) \leq C(\mathbf{u}) \leq \min(u_1, u_2, \ldots, u_n) = M^n(\mathbf{u}).
\]

5. The copula of any pair \((X, Y)\) of independent continuous random variables is \( IT^2(u, v) = uv \), \( IT^2 \) is called a product copula. Let \( \mathbf{C} \) be the copula of \((X_1, X_2, \ldots, X_n)\), \( n \geq 3 \). Then \( X_1, X_2, \ldots, X_n \) are pairwise independent if all \( \binom{n}{2} \) 2-margins of \( \mathbf{C} \) are \( IT^2 \); and \( X_1, X_2, \ldots, X_n \) are mutually independent if \( \mathbf{C}(\mathbf{u}) = u_1 u_2 \cdots u_n = IT^n(\mathbf{u}) \). Note that for continuous random variables, mutual independence implies pairwise independence. For more on copulas, see Nelsen (2006).

**Example 1.** Let \( \mathbf{C}_0(u_1, u_2, u_3) = u_1 u_2 u_3 [1 + \theta (1 - u_1)(1 - u_2)(1 - u_3)] \) for \( \theta \in [-1, 1] \) (a member of the Farlie–Gumbel–Morgenstern family of 3-copulas). If \( X_1, X_2, X_3 \) are continuous random variables with copula \( \mathbf{C}_0 \), then \( X_1, X_2, X_3 \) are pairwise independent for any \( \theta \) in \([-1, 1] \) but mutually independent only for \( \theta = 0 \).

The concept of a quasi-copula was introduced in Alsina et al. (1993) for the bivariate case, and Nelsen et al. (1996) for the general case, in order to characterize operations on distribution functions that can or cannot be derived from operations on random variables defined on the same probability space. In the last few years increasing attention has been devoted to these functions by researchers in some topics of fuzzy sets theory, such as preference modeling, similarities and fuzzy logics since quasi-copulas are a special type of aggregation function; see Beliakov et al. (2007) for an overview. An \( n \)-dimensional quasi-copula (for short, \( n \)-quasi-copula) is a function \( Q: [0, 1]^n \rightarrow [0, 1] \) which satisfies the boundary conditions (C1) and, instead of (C2), the following two conditions (Cuculescu and Theodorescu, 2001):

1. **Quasi-copula properties:** For every \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) and \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) in \([0, 1]^n \), then the Lipschitz condition \( |Q(\mathbf{u}) - Q(\mathbf{v})| \leq \sum_{i=1}^n |u_i - v_i| \) holds.

While every copula is an \( n \)-quasi-copula, there exist proper \( n \)-quasi-copulas, i.e., \( n \)-quasi-copulas which are not \( n \)-copulas (similarities of and differences between \( n \)-copulas and proper \( n \)-quasi-copulas can be found in De Baets et al., 2007, Fernández-Sánchez et al., 2011b, Nelsen et al., 2002, 2010) and Rodríguez-Lallena and Úbeda-Flores (2009). The pointwise best-possible bounds on nonempty sets of \( n \)-copulas are \( n \)-quasi-copulas (Fernández-Sánchez et al., 2011a; Nelsen and Úbeda-Flores, 2005; Nelsen et al., 2004; Rodríguez-Lallena and Úbeda-Flores, 2004a). Since \( W^n \) is not a copula whenever \( n \geq 3 \), it is a proper quasi-copula for such \( n \).
Averaging is used in several fields: voting, multicriteria and group decision making, statistical analysis, etc. An aggregation function \( f \) is said to be averaging if it is bounded by \( \min(u) \leq f(u) \leq \max(u) \) for every \( u \in [0, 1]^n \). The measure of orness \( \Omega \) is used to measure how far an averaging function \( f \) is from the max function, and is given by

\[
\Omega(f) = \frac{\int_{[0,1]^n} f(u)\,du - \int_{[0,1]^n} \min(u)\,du}{\int_{[0,1]^n} \max(u)\,du - \int_{[0,1]^n} \min(u)\,du}.
\]

See Beliaev et al. (2007) and the references therein. Since any \( n \)-quasi-copula \( Q \) is bounded by \( M^n \) and \( W^n \), we propose the following (multivariate) measure for \( Q \) based on the measure of orness:

\[
\xi_n(Q) = \frac{\int_{[0,1]^n} Q(u)\,du - \int_{[0,1]^n} W^n(u)\,du}{\int_{[0,1]^n} M^n(u)\,du - \int_{[0,1]^n} W^n(u)\,du}.
\]

So, basically, \( \xi_n \) measures how far a given \( n \)-quasi-copula \( Q \) is from \( W^n \). Note that \( \xi_n(M^n) = 1 \), \( \xi_n(W^n) = 0 \) and \( \xi_n(P^n) = [(n+1)! - 2^n]/[(n! - 1)2^n] \). Moreover, after some elementary algebra, \( \xi_n \) can be written in the following manner:

\[
\xi_n(Q) = \frac{(n+1)! \int_{[0,1]^n} Q(u)\,du - 1}{n! - 1}.
\]

Thus, given two \( n \)-quasi-copulas \( Q_1 \) and \( Q_2 \) such that \( Q_2(u) \leq Q_1(u) \) for all \( u \) in \([0, 1]^n\), \( \xi_n(Q_1) - \xi_n(Q_2) \) represents the normalized \( L_1 \)-distance between \( Q_1 \) and \( Q_2 \).

Finally, we observe that this measure is a function of the measure of orthant dependence \( \rho_n \), a multivariate version of the bivariate measure Spearman’s rho given in Nelsen (1996) applied to a quasi-copula \( Q \), i.e.,

\[
\rho_n(Q) = \frac{n + 1}{2^n - (n + 1)} \left( 2^n \int_{[0,1]^n} Q(u)\,du - 1 \right),
\]

in the following sense:

\[
\xi_n(Q) = \frac{(n+1)! - 2^n + n!(2^n - n - 1)\rho_n^{-1}(Q)}{2^n(n! - 1)}.
\]

3. The trivariate case

Let \( X_1, X_2, X_3 \) be continuous random variables with copula \( C \), and suppose that \( X_1, X_2, X_3 \) are pairwise independent. Are the best possible bounds on \( C \) the Fréchet–Hoeffding bounds, or are they other functions which are in some sense “closer together” than \( M^3 \) and \( W^3 \)?

We will answer this question by first finding the bounds when just one pair (say \( X_1 \) and \( X_2 \)) from \( X_1, X_2, X_3 \) are independent, then two pairs (say \( X_1 \) and \( X_2 \), and \( X_1 \) and \( X_3 \)), and finally all three pairs. To do so, we let \( C(12, 13, 23) \) denote the set of 3-copulas \( C(u_1, u_2, u_3) \) whose 2-margins are the given 2-copulas \( C_{12}(u_1, u_2), C_{13}(u_1, u_3), \) and \( C_{23}(u_2, u_3) \) (if \( C \) has fewer than three arguments, then fewer than three margins are specified); and consider bounds on the sets \( C(\Pi_{12}^2), C(\Pi_{23}^2), \) and \( C(\Pi_{12}^2, \Pi_{13}^2) \).

3.1. Bounds on \( C(\Pi_{12}^2) \)

Although bounds on sets of the form \( C(12), C(12, 13), \) and \( C(12, 13, 23) \) are known (Joe, 1997), it will be instructive to derive them for the cases where \( C_{ij} = \Pi_{ij}^2 \) (Deheuvels, 1983; Rüschendorf, 1991). Let \( C \) be a 3-copula in \( C(\Pi_{12}^2) \), that is, \( C \) is the copula of a triple \( X_1, X_2, X_3 \) when \( X_1 \) and \( X_2 \) are independent. Since \( C \) is nondecreasing in each variable, we have \( C(u_1, u_2, u_3) \leq C(u_1, u_2, 1) = u_1u_2 \) and \( C(u_1, u_2, u_3) \leq C(1, 1, u_3) = u_3 \). Hence \( C(u_1, u_2, u_3) \leq \min(u_1u_2, u_3) \). Since \( C \) is Lipschitz in \( u_3 \), we have \( C(u_1, u_2, 1) - C(u_1, u_2, u_3) \leq 1 - u_3 \), or equivalently \( C(u_1, u_2, u_3) \geq u_1u_2 + u_3 - 1 \). But we also have \( C(u_1, u_2, u_3) \geq 0 \), and hence \( C(u_1, u_2, u_3) \geq \max(u_1u_2 + u_3 - 1, 0) \). Thus, if \( C \) is in \( C(\Pi_{12}^2) \), then

\[
S_1(u) = \max(u_1u_2 + u_3 - 1, 0) \leq C(u) \leq \min(u_1u_2, u_3) = T_1(u).
\]

Neither \( S_1 \) nor \( T_1 \) is a copula, since \( V_{52}((3/1, 1) \times [1/2, 3/4] \times [2/3, 4/5]) = -1/20 = V_{31}([1/3, 1] \times [1/2, 3/4] \times [1/5, 1/3]) \); however, it is easy to show that \( W^3(u) \leq S_1(u) \leq \Pi^3(u) \leq T_1(u) \leq M^3(u) \). Furthermore, \( S_1 \) and \( T_1 \) are the pointwise best-possible bounds on the set \( C(\Pi_{12}^2) \). To establish this, we fix \((a, b, c)\) in \([0, 1]^3\) and construct two copulas \( C_{\xi} \) and \( C_{\gamma} \) in \( C(\Pi_{12}^2) \) such that \( C_{\xi}(a, b, c) = S_1(a, b, c) \) and \( C_{\gamma}(a, b, c) = T_1(a, b, c) \). \( C_{\xi} \) and \( C_{\gamma} \) are constructed by assigning probability mass \( V_{52}(B) \) and \( V_{52}(B) \) uniformly to each of five 3-boxes \( B \) in \([0, 1]^3 \) as shown in Table 1.

Therefore, \( S_1 \) and \( T_1 \) are proper quasi-copulas.

Now that we have pointwise best-possible bounds on the set \( C(\Pi_{12}^2) \), how much better are they than the Fréchet–Hoeffding bounds? For that, we use the measure \( \xi_3 \) defined in Section 2. Thus, we have \( \xi_3(T_1) = 11/15 \) and \( \xi_3(S_1) = 1/15 \). Consequently, the Fréchet–Hoeffding bounds have been narrowed by a factor of \( \xi_3(T_1) - \xi_3(S_1) = 2/3 \).
In two dimensions, the graph of $W^2$ is a “twisted reflection” in $\Pi^2$ of the graph of $M^2$, that is

$$M^2(u_1, u_2) - \Pi^2(u_1, u_2) = \Pi^2(u_1, 1 - u_2) - W^2(u_1, 1 - u_2).$$

An analogous relationship does not hold for $M^n$ and $W^n$ for any $n \geq 3$; but does hold for the functions $S_1(u)$ and $T_1(u)$:

$$T_1(u_1, u_2, u_3) - \Pi^3(u_1, u_2, u_3) = \Pi^3(u_1, u_2, 1 - u_3) - S_1(u_1, u_2, 1 - u_3).$$

Thus, in this sense, $S_1(u)$ and $T_1(u)$ are symmetric with respect to $\Pi^3(u)$.

### 3.2. Bounds on $C(\Pi^2_{12}, \Pi^2_{13})$

Let $C$ be a 3-copula in $C(\Pi^2_{12}, \Pi^2_{13})$, that is, $C$ is the copula of a triple $X_1, X_2, X_3$ when the two pairs $X_1, X_2$ and $X_1, X_3$ are independent. Since $C$ is nondecreasing in each variable, we have $C(u_1, u_2, u_3) \leq C(u_1, u_2, 1) = u_1 u_2$ and $C(u_1, u_2, u_3) \leq C(u_1, 1, u_3) = u_1 u_3$. Hence $C(u_1, u_2, u_3) \leq \min(u_1 u_2, u_1 u_3) = u_1 \min(u_2, u_3)$. Moreover, $V_C([0, u_1] \times [u_2, 1] \times [u_3, 1]) = u_1 - u_1 u_2 - u_1 u_3 + C(u_1, u_2, u_3) \geq 0$, or equivalently, $C(u_1, u_2, u_3) \geq (u_1 u_2 + u_1 u_3 - 1)$. Since $C(u_1, u_2, u_3) \geq 0$ we have $C(u_1, u_2, u_3) \geq u_1 \max(u_2 + u_3 - 1, 0)$. Thus, if $C$ is in $C(\Pi^2_{12}, \Pi^2_{13})$, then

$$S_2(u) = u_1 \max(u_2 + u_3 - 1, 0) \leq C(u) \leq u_1 \min(u_2, u_3) = T_2(u).$$

It is easy to show that $S_2$ and $T_2$ are copulas in $C(\Pi^2_{12}, \Pi^2_{13})$, and hence they are the pointwise best-possible bounds on $C(\Pi^2_{12}, \Pi^2_{13})$. To compare these bounds to the Fréchet–Hoeffding bounds (and to $S_1$ and $T_1$) we compute the measure $\xi_2$ for $S_2$ and $T_2$: $\xi_2(T_2) = 3/5$ and $\xi_2(S_2) = 1/5$. In this case the Fréchet–Hoeffding bounds have been narrowed by a factor of $\xi_2(T_2) - \xi_2(S_2) = 2/5$. Finally, we again have a “twisted reflection” symmetry for $S_2$ and $T_2$:

$$T_2(u_1, u_2, u_3) - \Pi^3(u_1, u_2, u_3) = \Pi^3(u_1, u_2, 1 - u_3) - S_2(u_1, u_2, 1 - u_3),$$

$$= \Pi^3(u_1, 1 - u_2, u_3) - S_2(u_1, 1 - u_2, u_3),$$

$$= \Pi^3(1 - u_1, u_2, u_3) - S_2(1 - u_1, u_2, u_3).$$

### 3.3. Bounds on $C(\Pi^2_{12}, \Pi^2_{13}, \Pi^2_{23})$

We now present the pointwise best-possible bounds on the copula of a triple $X_1, X_2, X_3$ when pairwise independent. The copula $C$ of $X_1, X_2, X_3$ is in $C(\Pi^2_{12}, \Pi^2_{13}, \Pi^2_{23})$. The bounds are computed by first calculating the C-volume of the eight 3-boxes of the form $\times^3_{i,j}[a_i, b_i]$ where each $[a_i, b_i]$ is either $[0, u_i]$ or $[u_i, 1]$; see Joe (1997) and Rodríguez-Lallena and Úbeda-Flores (2004b) for details. Then

$$S_3(u) \leq C(u) \leq T_3(u)$$

where

$$S_3(u) = \max(u_1(u_1 + u_2 - 1), u_2(u_1 + u_3 - 1), u_1(u_2 + u_3 - 1), 0),$$

$$T_3(u) = \min(u_1 u_2, u_2 u_3, u_1 u_3, (1 - u_1)(1 - u_2)(1 - u_3) + u_1 u_2 u_3).$$

Neither $S_3$ nor $T_3$ is a copula since $V_{S_3}(1/2, 2/3)^3 = -1/36 = V_{T_3}(1/3, 1/2)^3$; however, we again have $W^3(u) \leq S_3(u) \leq \Pi^3(u) \leq T_3(u) \leq M^3(u)$. Furthermore, $S_3$ and $T_3$ are pointwise best-possible and hence they are proper quasi-copulas; see Rodríguez-Lallena and Úbeda-Flores (2004b) for a construction similar to the one presented in Section 3.1. Again we compare these bounds to the Fréchet–Hoeffding bounds by computing the measure $\xi_3$ for $T_3$ and $S_3$: $\xi_3(T_3) = 49/100$ and $\xi_3(S_3) = 31/100$. In this case the Fréchet–Hoeffding bounds have been narrowed by a factor of $\xi_3(T_3) - \xi_3(S_3) = 9/50$ or 18%. Consequently, knowing that $X_1, X_2, X_3$ are pairwise independent forces the copula of $X_1, X_2, X_3$ considerably closer to the copula $\Pi^3$ of a mutually independent triple $X_1, X_2, X_3$. We also note the “twisted reflection” symmetry of $S_3$ and $T_3$:

$$T_3(u_1, u_2, u_3) - \Pi^3(u_1, u_2, u_3) = \Pi^3(u_1, u_2, 1 - u_3) - S_3(u_1, u_2, 1 - u_3),$$

$$= \Pi^3(u_1, 1 - u_2, u_3) - S_3(u_1, 1 - u_2, u_3),$$

$$= \Pi^3(1 - u_1, u_2, u_3) - S_3(1 - u_1, u_2, u_3).$$

### Table 1

<table>
<thead>
<tr>
<th>$B$</th>
<th>$V_C(B)$</th>
<th>$\Pi^2(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0, a] \times [0, b] \times [0, c]$</td>
<td>$\max(ab + c - 1, 0)$</td>
<td>$\min(ab, c)$</td>
</tr>
<tr>
<td>$[0, a] \times [0, b] \times [c, 1]$</td>
<td>$\min(1 - c, ab)$</td>
<td>$\max(ab - c, 0)$</td>
</tr>
<tr>
<td>$[a, 1] \times [0, b] \times [0, 1]$</td>
<td>$b(1 - a)$</td>
<td>$b(1 - a)$</td>
</tr>
<tr>
<td>$[0, a] \times [b, 1] \times [0, 1]$</td>
<td>$a(1 - b)$</td>
<td>$a(1 - b)$</td>
</tr>
<tr>
<td>$[a, 1] \times [b, 1] \times [0, 1]$</td>
<td>$(1 - a)(1 - b)$</td>
<td>$(1 - a)(1 - b)$</td>
</tr>
</tbody>
</table>
4. The n-variate case

We now consider a vector \( X = (X_1, X_2, \ldots, X_n) \) of real-valued continuous random variables which are pairwise independent; that is, the copula \( C \) of \( X \) is a member of the set \( \mathcal{C}(\Pi^2) \) \( 1 \leq i < j \leq n \), i.e., all \( \binom{n}{2} \) 2-margins of \( C \) are \( \Pi^2 \). Since \( C \) is nondecreasing in each variable, it follows that \( C \leq T_n \), where

\[
T_n(u) = \min(u, u_j \mid 1 \leq i < j \leq n).
\]

Although \( T_n \) is neither a copula (but rather a quasi-copula, as can be easily proved) nor a pointwise best-possible upper bound for \( C \), it will suffice for our purposes. First note that

\[
\int_{[0,1]^n} T_n(u) \, du = \frac{3}{(n+1)(n+2)},
\]

whence

\[
\xi_n(T_n) = \frac{n+2}{n+1} - \frac{1}{n+1}.
\]

For any \( u \in [0, 1]^n \), we have by the triangle inequality

\[
|C(u) - \Pi^n(u)| \leq |C(u) - W^n(u)| + |W^n(u) - \Pi^n(u)| = |C(u) - W^n(u)| + |\Pi^n(u) - W^n(u)|.
\]

Since \( W^n \leq C \leq T_n \leq M^n \) and \( W^n \leq \Pi^n \leq T_n \leq M^n \), we obtain

\[
\int_{[0,1]^n} |C(u) - \Pi^n(u)| \, du \leq \int_{[0,1]^n} W^n(u) \, du - \int_{[0,1]^n} W^n(u) \, du = \left[ \xi_n(C) + \xi_n(\Pi^n) \right] \leq 2 \xi_n(T_n).
\]

Since \( \lim_{n \to \infty} \xi_n(T_n) = 0 \), the normalized \( L_1 \)-distance between the copula \( C \) of a vector \( X \) of pairwise independent random variables and the copula \( \Pi^n \) of the corresponding vector of mutually independent random variables approaches 0 as the dimension increases.

5. Pairwise and mutual exchangeability

A natural question is the following: Is there a relationship between pairwise and mutual exchangeability analogous to the one between pairwise and mutual independence?

Recall that a pair \( (X, Y) \) of random variables are exchangeable if \( (X, Y) \) has the same distribution as \( (Y, X) \). For identically distributed continuous random variables, this is equivalent to the symmetry of the copula \( C \) of \( (X, Y) \), i.e., \( C(u, v) = C(v, u) \) for all \( u, v \) in \([0, 1]\). Similarly, a vector \( X = (X_1, X_2, \ldots, X_n) \) of random variables is (mutually) exchangeable if for any permutation \( \sigma \) of \( \{1, 2, \ldots, n\} \), the vector \( (X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}) \) has the same distribution as \( X \).

Consequently, if the elements of \( X \) are continuous and identically distributed, pairwise and mutual exchangeability of \( X \) correspond, respectively, to symmetry of the 2-margins of \( C \) and permutation symmetry of \( C \), i.e., \( C(u) = C(u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(n)}) \) for any permutation \( \sigma \) of \( \{1, 2, \ldots, n\} \).

The following example demonstrates the existence of copulas with symmetric 2-margins that are not permutation symmetric.

**Example 2.** Let \( C \) be a symmetric 2-copula, and set \( C'(u_1, u_2, u_3) = C(u_1, \min(u_2, u_3)) \). \( C' \) is a 3-copula (Quesada-Molina and Rodríguez-Lallena, 1994) whose margins are symmetric, but \( C' \) is permutation symmetric if and only if \( C = M^2 \). The 3-copula \( T_2 \) in Section 3.2 is the member of this family with \( C = \Pi^2 \).

Since permutation symmetric \( n \)-copolas are found throughout the set of \( n \)-copolas (e.g., any convex linear combination of \( M^n \) and \( \Pi^n \) is permutation symmetric), calculating bounds on the set of \( n \)-copolas with symmetric 2-margins is not an effective means of measuring how “close” pairwise exchangeability is to permutation exchangeability. If \( C \) is a member of \( \{C(u) \mid \mathcal{C}_0(u_1, u_2) = C_0(u_1, u_2), 1 \leq i < j \leq n \} \) and \( S_n', T_n' \) are lower and upper bounds for \( C \) (note that \( T_n' = M^n \)), then

\[
\xi_n(S_n') - \xi_n(T_n') \geq \xi_n(M^n) - \xi_n(\Pi^n) = \frac{1}{1 - \frac{1}{n}},
\]

and consequently \( \lim_{n \to \infty} \left[ \xi_n(T_n') - \xi_n(S_n') \right] = 1 \).

Finally, if \( X = (X_1, X_2, \ldots, X_n) \) is both pairwise independent and mutually exchangeable, must \( X \) be mutually independent? That is, if \( C \) is the copula of \( X \), and \( C \) is permutation symmetric and all the 2-margins of \( C \) are \( \Pi^2 \), must \( C \) equal \( \Pi^n \) ? The answer is no for every \( n \geq 3 \), as the following counterexample demonstrates.
Example 3. The copulas in Example 1 can be generalized to the following family of \( n \)-copulas for any \( n \geq 3 \):

\[
C_\theta(u) = u_1 u_2 \cdots u_n [1 + \theta (1 - u_1) (1 - u_2) \cdots (1 - u_n)], \quad \theta \in [-1, 1],
\]

which is readily seen to be permutation symmetric with all 2-margins equal to \( \Pi^2 \) (indeed, all \( k \)-margins are \( \Pi^k \) for \( 2 \leq k < n \)); and \( C_\theta = \Pi^n \) if and only if \( \theta = 0 \).

However, an infinite sequence of pairwise independent exchangeable random variables are mutually independent. See Hu (1997) for details.

Acknowledgments

The authors thank the reviewer for comments on an earlier version. The second author acknowledges the support of the Ministerio de Ciencia e Innovación (Spain) and FEDER, under research project MTM2009-08724.

References


