Proof Without Words: Nested Square Roots

Roger B. Nelsen (nelsen@lclark.edu), Lewis & Clark College, Portland, OR

Many infinite nested square root expressions have the form

\[ x = \sqrt{a + b\sqrt{a + b\sqrt{a + \cdots}}} \]

for \(a\) and \(b\) positive and can be evaluated by observing that \(x = \sqrt{a + bx}\), squaring to obtain \(x^2 = a + bx\), and solving for the positive root. An alternative method begins by dividing the quadratic by \(x\) to obtain \(x = b + a/x\).

**Theorem.** For \(a, b\) positive, \(x = \sqrt{a + b\sqrt{a + b\sqrt{a + \cdots}}} = \frac{1}{2} \left( b + \sqrt{b^2 + 4a} \right)\).

**Proof.** Since \(x = b + a/x\),

\[
\begin{align*}
(2x - b)^2 &= b^2 + 4a, \\
2x^2 - 2bx - b^2 &= b^2 + 4a, \\
x^2 - bx - 2b &= 2a, \\
x^2 - bx - 2b &= 2a, \\
\end{align*}
\]

so that \((2x - b)^2 = b^2 + 4a\).

**Corollary.**

\[
\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}} = 3 = \sqrt{3 + 2\sqrt{3 + 2\sqrt{3 + \cdots}}},
\]

\[
\sqrt{12 + \sqrt{12 + \sqrt{12 + \cdots}}} = 4 = \sqrt{4 + 3\sqrt{4 + 3\sqrt{4 + \cdots}}},
\]

\[
\sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}} = 5 = \sqrt{5 + 4\sqrt{5 + 4\sqrt{5 + \cdots}}},
\]

**Proof.** In the theorem, for \(n \geq 3\), set \((a, b) = (n(n - 1), 1)\) and \((n, n - 1)\).

**Summary.** We evaluate some nested square roots by computing the area of a square in two ways.

http://dx.doi.org/10.4169/college.math.j.48.3.204
MSC: 97F50

© THE MATHEMATICAL ASSOCIATION OF AMERICA