Proof Without Words: Perfect Numbers Modulo 7
Roger B. Nelsen (nelsen@lclark.edu), Lewis & Clark College, Portland, OR

Theorem ([1]). Every even perfect number \( N_p = 2^{p-1}(2^p - 1) \) for prime \( p \neq 3 \) is congruent to 1 or 6 modulo 7. In particular,

\[ p \equiv 1 \text{ mod } 3 \implies N_p \equiv 1 \text{ mod } 7 \quad \text{and} \quad p \equiv 2 \text{ mod } 3 \implies N_p \equiv 6 \text{ mod } 7. \]

Proof. \( N_p = 2^{p-1}(2^p - 1) = T_{2^{p-1}} \) where \( T_n = 1 + 2 + \cdots + n = n(n + 1)/2 \) is the \( n \)th triangular number:

\[
\begin{align*}
2^{p-1} & \quad 2^{p-1} - 1 \\
2^{p-1} & \quad 2^{p-1}
\end{align*}
\]

\[ p = 3k + 1 \implies 2^p - 1 = 2 \cdot 8^k - 1 \equiv 1 \text{ mod } 7 \implies N_{3k+1} = T_{7n+1}, \]

\[ p = 3k + 2 \implies 2^p - 1 = 4 \cdot 8^k - 1 \equiv 3 \text{ mod } 7 \implies N_{3k+2} = T_{7n+3}. \]

\[
\begin{align*}
T_{7n+1} & = 35T_n + 14T_{n-1} + 1, \\
\text{so } N_{3k+1} & \equiv 1 \text{ mod } 7. \\
T_{7n+3} & = 49T_n + 6, \\
\text{so } N_{3k+2} & \equiv 6 \text{ mod } 7.
\end{align*}
\]

Summary. We partition triangular numbers to show wordlessly that every even perfect number except 28 is congruent to 1 or 6 modulo 7.

Reference

http://dx.doi.org/10.4169/college.math.j.48.1.17
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