While looking through some elementary number theory texts recently, I was struck by how few illustrations most of them have. A number can represent the cardinality of a set, the length of a line segment, or the area of a plane region, and such representations naturally lead to a variety of visual arguments for topics in elementary number theory. Since a course in number theory usually begins with properties of the integers, specifically the positive integers, the texts should have more pictures. In the next few pages I’ll present some “visual gems” illustrating one-to-one correspondence, congruence, Pythagorean triples, perfect numbers, and the irrationality of the square root of two.

We begin with *figurate numbers*, positive integers that can be represented by geometric patterns. The simplest examples are the triangular numbers \( t_n = 1 + 2 + 3 + \cdots + n \) and square numbers \( n^2 \). In Figure 1(a) are the familiar patterns for \( t_4 \) and \( 4^2 \). The computational formula for \( t_n \) is \( n(n+1)/2 \), which is easily derived from Figure 1(b) by seeing that \( 2t_n = n(n+1) \).

![Figure 1](image1.png)

**Figure 1.** Triangular and square numbers.

You may have noticed that each triangular number is also a binomial coefficient:

\[
1 + 2 + \cdots + n = \binom{n+1}{2}.
\]

One explanation for this is that each is equal to \( n(n+1)/2 \), but this answer sheds no light on why the relationship holds. A better explanation is the following: there exists a one-to-one correspondence between a set of \( t_n = 1+2+3+\cdots+n \) objects and the set of two-element subsets of a set with \( n+1 \) objects. In Figure 2, we have a visual proof of this correspondence.

![Figure 2](image2.png)

**Figure 2.** A visual proof that \( 1 + 2 + \cdots + n \) equals \( \binom{n+1}{2} \).

There are many lovely relationships between triangular and square numbers. One is hidden in the following theorem, which appears in nearly every elementary number theory text as an example or exercise: if \( n \) is odd, then \( n^2 \equiv 1 \pmod{8} \). In Figure 3, we have a visual proof that every odd square is one more than eight times a triangular number, which proves the result. But it does more—it is an essential part of an induction proof that there are infinitely many numbers that are simultaneously square and triangular! First note that \( t_1 = 1 = 1^2 \). Then observe that

\[
t_{8t_k} = \frac{8t_k (8t_k + 1)}{2} = 4t_k (2k + 1)^2,
\]

so that if \( t_k \) is a square, so is \( t_{8t_k} \). For example, \( t_8 = 6^2, t_{288} = 204^2 \), etc. But not all square triangular numbers are generated this way: \( t_{49} = 35^2, t_{1681} = 1189^2 \), etc.

**Challenge:** Can you find a visual argument to use in the proof for odd triangular numbers?
Congruence results involving triangular numbers can be illustrated in a similar manner. In Figure 4, we “see” that \( t_{3n-1} \equiv 0 \pmod{3} \).

**Exercise:** Find visual arguments to show that \( t_{3n} \equiv 0 \pmod{3} \) and \( t_{3n+1} \equiv 1 \pmod{3} \).

**Figure 4.** A visual proof that \( t_{3n-1} \equiv 0 \pmod{3} \).

Discussing triangles and squares in geometry always brings to mind the Pythagorean theorem. In number theory it brings to mind *Pythagorean triples*. A primitive Pythagorean triple (PPT) is a triple \((a, b, c)\) of positive integers with no common factors satisfying \( a^2 + b^2 = c^2 \). Familiar examples are \((3, 4, 5)\), \((5, 12, 13)\) and \((8, 15, 17)\). Indeed, there are infinitely many PPTs, and in Figure 5 we have a visual proof that \( n^2 \text{ odd} \) (i.e. \( n^2 = 2k+1 \)) implies that \( n^2 + k^2 = (k+1)^2 \), or equivalently, that \((n, k, k+1)\) is a PPT. In Figure 5, the \( n^2 = 2k+1 \) blue dots are first arranged into an “L” then \( k^2 \) white dots are drawn, for a total of \((k+1)^2\) dots.

**Exercise:** In the proof, we’ve actually shown that there are infinitely many PPTs in which one leg and the hypotenuse are consecutive integers. Find a visual proof that there are infinitely many PPTs in which the hypotenuse and one leg differ by two.

**Figure 5.** There are infinitely many PPTs.

A classical result characterizing PPTs is the following theorem: \((a, b, c)\) is a PPT if and only if there are relatively prime integers \(m > n > 0\), one even and the other odd, such that (\(a, b, c\) or \(b, a, c\)) = \((2mn, m^2 - n^2, m^2 + n^2)\). Figure 6 uses the double angle formulas from trigonometry to illustrate this result.

**Figure 6.** A characterization of PPTs.

Another less well-known characterization is the following: there exists a one-to-one correspondence between PPTs and factorizations of even squares of the form \( n^2 = 2pq \) with \( p \) and \( q \) relatively prime. Figure 7 illustrates this result.

**Figure 7.** Another characterization of PPTs.
The PPT (3, 4, 5) is the only such triple where the sum of two consecutive squares is the next square. But what about sums of three consecutive squares? Four consecutive squares? Observe:

\[
\begin{align*}
3^2 + 4^2 &= 5^2 \\
10^2 + 11^2 + 12^2 &= 13^2 + 14^2 \\
21^2 + 22^2 + 23^2 + 24^2 &= 25^2 + 26^2 + 27^2
\end{align*}
\]

What’s the pattern? A little reflection yields the observation that each square immediately to the left of the equals sign is the square of four times a triangular number, i.e., \(4 = 4(1), 12 = 4(1+2), 24 = 4(1+2+3)\), etc., which leads to the identity

\[
(4t_n - n)^2 + \cdots + (4t_n)^2 = (4t_n + 1)^2 + \cdots + (4t_n + n)^2.
\]

While it is not difficult to verify this by induction, the illustration (for the \(n = 3\) case, using \(4t_3 = 4(1+2+3)\)) of these “Pythagorean runs” in Figure 8 may better explain why the relationship holds.

The ancient Greeks called positive integers like 6 and 28 perfect, because in each case the number is equal to the sum of its proper divisors (e.g., \(28 = 1 + 2 + 4 + 7 + 14\)). In Book IX of the Elements, Euclid gives a formula for generating even perfect numbers: If \(p = 2^{n-1} - 1\) is a prime number, then \(N = 2^n p\) is perfect. We can illustrate this result very nicely in Figure 9 by letting \(N\) be the area of a rectangle of dimensions \(2^n \times p\), and noting that the rectangle can be partitioned into a union of smaller rectangles (and one square) whose areas are precisely the proper divisors of \(N\).

We conclude with one more example from the Greeks. The Pythagoreans were probably the first to demonstrate that \(\sqrt{2}\) is irrational, that is, to show that lengths of a side and the diagonal of a square are incommensurable. Here is a modern version of that classical Greek proof. From the Pythagorean theorem, an isosceles triangle of edge-length 1 has hypotenuse \(\sqrt{2}\). If \(\sqrt{2}\) is rational, then some multiple of this triangle must have three sides with integer lengths, and hence there must be a smallest isosceles right triangle with this property. But the difference of two segments of integer length is a segment of integer length and so, as illustrated in Figure 10, inside any isosceles right triangle whose three sides have integer lengths we can always construct a smaller one with the same property! Hence \(\sqrt{2}\) must be irrational.

**Exercise:** Prove that \(\sqrt{3}\) and \(\sqrt{5}\) are irrational in a similar manner.

**Further Reading**

For more visual proofs with triangular numbers, see James Tanton’s “Triangular Numbers” in the November 2005 issue of Math Horizons. For visual proofs in combinatorics, see Jennifer Quinn and Arthur Benjamin’s Proofs That Really Count: The Art of Combinatorial Proof (MAA, 2003). For an

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square identities and integrals. Thus, when approaching problems of this nature, it is often best to represent numbers as areas of a region of a plane. For instance, a simple figure (such as the one on top of page 7) represents that \(1 + 3 + 5 + \cdots + (2n-1) = n^2\), even though the algebraic proof is a better-known method.

There are many ways to solve a math problem, and visual approaches are often a great start. Because of the nature of visual proofs, it is possible to make different illustrations to arrive at the same conclusion. For example, the authors continually revisit the Pythagorean Theorem, showing several visual proofs throughout the book to demonstrate that there are various ways to arrive at the same conclusion. From the ancient Greeks’ to Leonardo da Vinci’s method, each proof is different as well as provocative for such a fundamental theorem. Other topics covered include iterative procedures, some number theory, visual sets, and a mind-bendingly fun chapter on three-dimensional illustration proofs.

As a narrative, Math Made Visual is truly a work of art. However, the best feature in Math Made Visual is the problem sets at the end of each chapter. Each puzzle is challenging and fun, and, just like the rest of the book, invites creativity. Although there are solutions for each problem in the back of the book, it is quite easy to arrive at the same conclusion in other ways. Regardless, each solution is elegant as well as interesting.

Having double majored in art and math, I could not help delighting in Math Made Visual for its true spirit and meaning. Although useful and fun, the aesthetics of each visual proof was appealing as well as entertaining. Ultimately, each treasure found in unraveling a wordless proof exposes a deeper and insightful glimpse of the beautiful processes of mathematics.


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Some of the proofs in the article originally came from:

D. Goldberg, personal communication.

Contest Results

In the April 2007 issue of Math Horizons, Ken Suman challenged our readers to find examples of mathematical equivoques. Many of you responded. Our favorites come from Benjamin Dickman, Amherst College ’08 and Donald Vestal, South Dakota State University with these new definitions:

concave: where the bad guys hang out
convex: what police try to do
dimension: “Did I tell you...”
fuzzy set: A set that has been left in the refrigerator too long
mock theta functions: what Ken Ono does when his theta functions misbehave
Q Test: most adorable
relativity: genealogy
Schur numbers: numbers whose values are known
Schur triple: Catching the right-fielder asleep with a hit along the base line

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