Periodic Continued Fractions Via a Proof Without Words

Roger B. Nelsen

To cite this article: Roger B. Nelsen (2018) Periodic Continued Fractions Via a Proof Without Words, Mathematics Magazine, 91:5, 364-365, DOI: 10.1080/0025570X.2018.1456151

To link to this article: https://doi.org/10.1080/0025570X.2018.1456151

Published online: 07 Dec 2018.
An infinite simple continued fraction is an expression of the form

\[ [a_0, a_1, a_2, a_3, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \]

where \(a_0\) is an integer and \(a_1, a_2, a_3, \ldots\) are positive integers. When a block of \(a_k\)'s repeats over and over, the continued fraction is periodic. For example,

\[ [\overline{a}] = [a, a, a, \ldots] = a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \cdots}}} \]

and \([\overline{a, b}] = [a, b, a, b, \ldots] = a + \frac{1}{b + \frac{1}{a + \frac{1}{b + \cdots}}} \]

are periodic (just like in repeating decimals, the vinculum indicates the repeating block). We find a closed form expression for \([\overline{a, b}]\) (note that \([\overline{a}]\) is a special case of \([\overline{a, b}]\)) by proving Lemma 1 wordlessly with a frequently used diagram [1]-[4], proving Lemma 2 by simple algebra, from which the theorem follows immediately.

Lemma 1. If \(x > 0\) and \(x = a + \frac{a}{bx}\), then \(x = \frac{1}{2} [a + \sqrt{a^2 + 4(a/b)}]\).

Proof.

\[
\begin{aligned}
x + a/bx &= 2x - a \\
(2x - a)^2 &= a^2 + 4(a/b) \\
x &= \frac{1}{2} \left[a + \sqrt{a^2 + 4(a/b)}\right]
\end{aligned}
\]

Lemma 2. If \(x = [\overline{a, b}] = a + \frac{1}{b + \frac{1}{x}}\) then \(x = a + \frac{a}{bx}\).

Proof. Because \(x = a + \frac{1}{b + \frac{1}{x}}\), then \(x\left(b + \frac{1}{x}\right) = a\left(b + \frac{1}{x}\right) + 1\). It follows that \(bx = ab + \frac{a}{x}\) so that \(x = a + \frac{a}{bx}\). □
Theorem. The periodic continued fraction \([a, b]\) equals \(\frac{1}{2}[a + \sqrt{a^2 + 4(a/b)}]\).

As examples, notice that \([1] = \frac{1}{2}(1 + \sqrt{5})\), \([2] = 1 + \sqrt{2}\), \([2, 1] = 1 + \sqrt{3}\), and \([3, 2] = \frac{1}{2}(3 + \sqrt{15})\).

Exercise. Show that for \(n\) a positive integer,

(a) \([n, 2n] = \sqrt{n^2 + 1}\)  
(b) \([n, n, 2n] = \sqrt{n^2 + 2}\).

(c) \([n, 2, 2n] = \sqrt{n^2 + n}\), and  
(d) \([n, 1, 2n] = \sqrt{n^2 + 2n}\).

As a hint, consider \([2n], [2n, n], [2n, 2]\), and \([2n, 1]\).

Acknowledgment The author wishes to thank two referees and the Editor for helpful suggestions on an earlier draft of this note.

REFERENCES


Summary. We evaluate some periodic continued fractions by computing the area of a square in two different ways.

ROGER B. NELSEN (MR Author ID: 237909) is a professor at Lewis & Clark College, where he taught mathematics and statistics for 40 years.