On Candido’s Identity

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Giacomo Candido [1] (1871–1941) proved the equality

\[ [F_n^2 + F_{n+1}^2 + F_{n+2}^2]^2 = 2[F_n^4 + F_{n+1}^4 + F_{n+2}^4]. \]

where \( F_n \) denotes the \( n \)th Fibonacci number, by observing that for all reals \( x, y \) one has the curious identity

\[ [x^2 + y^2 + (x + y)^2]^2 = [x^4 + y^4 + (x + y)^4]. \] (1)

Candido’s identity (1) can be easily shown to be true not only in \( \mathbb{R}^+ := [0, \infty) \) but also in any commutative ring and admits a clear visual description as presented recently in [3]. This identity raises the question: is (1) a characteristic property of the polynomial function \( y = x^2 \) in \( \mathbb{R}^+ \)? In order to answer this we reformulate (1) as follows. Let \( f \) be a function from \( \mathbb{R}^+ \) into \( \mathbb{R}^+ \) such that

\[ f(f(x) + f(y) + f(x + y)) = 2[f(f(x)) + f(f(y)) + f(f(x + y))]. \] (2)

In general (2) admits trivial solutions like \( f \equiv 0 \) as well as many bizarre, highly discontinuous solutions. For example, define \( f \) to be any function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \) with the property that \( f(x) = 0 \) whenever \( x \) is rational and \( f(x) \) is rational (but arbitrary!) whenever \( x \) is irrational. It is an exercise (try it) to show that every possible combination of rational or irrational values for the inputs \( x \) and \( y \) reduces (2) to the identity 0 = 0. But if we require \( f \) to be a continuous surjection on \( \mathbb{R}^+ \) with \( f(0) = 0 \), then we shall show that \( f \) can differ from the squaring function only by a multiplicative constant.

**Lemma.** For any two positive real numbers \( a \) and \( b \) with \( 0 < a < b \), there are integers \( m \) and \( n \) such that \( a < 2^m 3^n < b \).

**Proof.** We consider three cases.

Case 1. If \( 1 \leq a < b \) then \( 0 \leq \log_2(a) < \log_2(b) \) and it follows that \( \log_2(a)/3^n < \log_2(b)/3^n < 1 \) for a sufficiently large positive integer \( n \). Since \( 2^p \neq 3^q \) for all integers \( p, q \) such that \( p, q \neq 0 \), we deduce \( p \log 2 \neq q \log 3 \), i.e., \( \log_2(3) = \log 3/\log 2 \) is clearly irrational (see, e.g., [2]). So it follows from the equidistribution theorem [4,
Theorem 6.2, p. 72] that the sequence $\log_2(3), 2 \log_2(3), 3 \log_2(3), \ldots$ is uniformly distributed modulo 1, i.e., there is some positive integer $m$ such that

$$\log_2(a)/3^n < \log_2(3^m) - \lfloor \log_2(3^m) \rfloor < \log_2(b)/3^n,$$

where $\lfloor x \rfloor$ denotes the greatest integer $k \leq x$. Let $r = \log_2(3^m)$ and let $s = r - \lfloor r \rfloor$. Then since $2^r = 3^m$, it follows that $2^s = 3^m/2^{\lfloor r \rfloor}$. With this notation

$$\log_2(a) < 3^n s < \log_2(b)$$

i.e., $a < 2^{(3^n s)} < b$, whence $a < (3^m/2^{\lfloor r \rfloor})3^n < b$. This shows that there is an integral power of 2 times an integral power of 3 between $a$ and $b$.

Case 2. If $a < 1 < b$ we can use $n = m = 0$.

Case 3. If $0 < a < b \leq 1$ we will have $1 \leq 1/b < 1/a$ so by case 1 there exist integers $m, n$ such that $1/b < 2^m 3^n < 1/a$ and therefore $a < 2^{-m} 3^{-n} < b$.

Now we prove the following:

**Theorem.** A continuous surjective function $f$ from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $f(0) = 0$ satisfies Candido’s equation (2) if and only if

$$f(x) = kx^2,$$

where $k > 0$ is an arbitrary constant.

**Proof.** From Candido’s equality (1), it follows that (3) satisfies (2). Conversely, assume that $f$ is a solution of (2) satisfying the above conditions. Since $f(0) = 0$ the substitution $y = 0$ into (2) yields that for all $x \geq 0$: $f(2f(x)) = 4f(f(x))$. Since $f$ is surjective, $f(x)$ ranges throughout $\mathbb{R}^+$ as $x$ ranges throughout $\mathbb{R}^+$, so that if we let $z = f(x)$, we have $f(2z) = 4f(z)$ for all $z$ in $\mathbb{R}^+$. It follows by induction

$$f(2^n z) = (2^n)^2 f(z),$$

for all integers $n \geq 0$.

Since $f(z) = f(2^n (z/2^n)) = (2^n)^2 f(z/2^n)$ we get

$$f(2^{-n} z) = (2^{-n})^2 f(z)$$

for all integers $n \geq 1$. Thus from (4) and (5) we can conclude

$$f(2^n z) = (2^n)^2 f(z),$$

for all integers $n$. Next, set $y = x$ in (2) to obtain

$$f(2f(x) + f(2x)) = 4f(f(x)) + 2f(f(2x)),$$

and by virtue of (6), using $f(2x) = 4f(x)$, we get:

$$4f(3f(x)) = f(6f(x)) = 4f(f(x)) + 2 \cdot 4^2 \cdot f(f(x)) = 36f(f(x)),$$

i.e., with $f(x) = z \geq 0$ arbitrary, $f(3z) = 3^2 f(z)$ and by induction $f(3^m z) = (3^m)^2 f(z)$, whenever $m \geq 0$. As above, $f(z) = f(3^m (z/3^m)) = (3^m)^2 f(z/3^m)$ so $f(3^{-m} z) = (3^{-m})^2 f(z)$ and therefore

$$f(3^m z) = (3^m)^2 f(z),$$
for all integers \( m \). By means of (6) and (7), we obtain that for all integers \( m, n \):

\[
f(2^n 3^m) = (2^n 3^m)^2 f(1).
\] (8)

By our previous lemma any real numbers in \([0, \infty)\) may be approximated by a sequence in the set \( \{2^n 3^m | n, m \text{ integers}\} \) so from (8) and the continuity of \( f \) we can conclude that for all \( x \) in \( \mathbb{R}^+ \), \( f(x) = kx^2 \), with \( k = f(1) > 0 \) an arbitrary constant.

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**Monotonic Convergence to \( e \) via the Arithmetic-Geometric Mean**

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Recently, Hansheng Yang and Heng Yang [3], by using only the arithmetic-geometric inequality, have proved the monotonicity of the sequences \( (x_n) \), \( (y_n) \), related to the number \( e \):

\[
x_n = \left(1 + \frac{1}{n}\right)^n, \quad y_n = \left(1 + \frac{1}{n}\right)^{n+1} \quad (n = 1, 2, \ldots)
\]

Such a method probably is an old one and has been applied e.g. in [1], or [2].

We want to show that the above monotonicities can be proved much easier than in [3].

Recall that the arithmetic-geometric inequality says that for \( a_1, \ldots, a_k > 0 \), and

\[
G_k = G_k(a_1, \ldots, a_k) = \sqrt[n]{a_1 \ldots a_k},
\]

\[
A_k = A_k(a_1, \ldots, a_k) = \frac{a_1 + \cdots + a_k}{k},
\]

we have

\[
G_k \leq A_k, \quad (1)
\]

with equality only when all \( a_i \) are equal.