Proof Without Words: Square Triangular Numbers and Almost Isosceles Pythagorean Triples

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**Theorem (1, §4.9).** Let \( T_n = 1 + 2 + \cdots + n = n(n + 1)/2 \) denote the \( n \)th triangular number. Then
\[
T_n = k^2 \text{ if and only if } (2n + 2k + 1)^2 = (n + 2k)^2 + (n + 2k + 1)^2.
\]

*Proof.* (Using inclusion-exclusion, shown for \((n, k) = (8, 6)\).)

\[
(2n + 2k + 1)^2 = (n + 2k)^2 + (n + 2k + 1)^2 - (2k)^2 + 2n(n + 1),
\]

\[
(2n + 2k + 1)^2 = (n + 2k)^2 + (n + 2k + 1)^2 \iff 4k^2 = 4T_n.
\]

**Summary.** We illustrate wordlessly a one-to-one correspondence between square triangular numbers and almost isosceles Pythagorean triples.

**References**


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